1 Mathematics 433-A **Fall 2000**

1.1 **Exam 2**

November 21, 2000

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Name

Technology used: ____

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. (6 points each) Give the definitions of the following.
   (a) The **product group** $G \times G'$ of two groups G and G'.
   (b) The **quotient group** G/K of a group G by a normal subgroup K. Be sure to indicate the binaray operation in G/K.
   (c) The **orbit** of an element $s \in S$ where G is a group acting on the set S.
   (d) The **stabilizer** of an element $s \in S$ where G is a group acting on the set S.
   (e) A **rigid motion** of the plane to itself.

2. (10 points each) If G is a group acting on the set S, the element s is arbitrary in S, and G_s is the stabilizer of s in G, then there is a map from the coset space of G_s in G to the orbit of s defined by

$$\begin{aligned} \phi &: G/G_s \to O_s \\ \phi(aH) &= as \end{aligned}$$

   Prove that this map ϕ is
   (a) one-to-one
   (b) onto

3. (15 points) Use a group action to count the rotational symmetries of a cube. Be explicit about what you choose as your set S.

4. (10 points) Do **one** of the following.
   (a) Prove if $|G| = p$ where p is a prime number, then G is isomorphic to a cyclic group of order p.
   (b) Determine all automorphisms of the group C_4. Be sure to show your functions are automorphisms.

5. (15 points) Do **one** of the following.
   (a) Let G be a subgroup of M that contains rotations by $\theta = \pi$ about two points: the origin and the point with coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Prove **algebraically** that G contains a translation. [See the Useful Facts at the end of the examination for tools.]

(b) Show **algebraically** that the successive reflection across two different lines through the origin is a rotation. For your proof, use the specific lines that form angles of $\pi/4$ and $\pi/2$ with the positive x_1- axis. What is the angle θ for the resulting rotation ρ_θ? [See the Useful Facts at the end of the examination for tools.]

6. (10 points) Find all matrices in the stabilizer of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ if the group action is conjugation in $GL(2, R)$. A useful fact is that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

1.2 Useful Facts

- **Theorem 1** *Every rigid motion can be written in one of the forms (uniquely)* $m = t_a\rho_\theta$ *or* $m = t_a\rho_\theta r$ *by using the following formulas for composition.*

  1. $t_a t_b = t_{a+b}$
  2. $\rho_\theta \rho_\eta = \rho_{\theta+\eta}$
  3. $rr = i$
  4. $\rho_\theta t_a = t_{a'} \rho_\theta$, *where* $a' = \rho_\theta (a)$
  5. $r t_a = t'_a r$, *where* $a' = r(a)$
  6. $r\rho_\theta = \rho_{-\theta} r$.